

MACFARLANE HYPERBOLIC 3-MANIFOLDS

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ABSTRACT. We identify and study a class of hyperbolic 3-manifolds (which we call Macfarlane manifolds) whose quaternion algebras admit a geometric interpretation analogous to Hamilton's classical model for Euclidean rotations. We characterize these manifolds arithmetically, and show that infinitely many commensurability classes of them arise in diverse topological and arithmetic settings. We then use this perspective to introduce a new method for computing their Dirichlet domains. We also give similar results for a class of hyperbolic surfaces and explore their occurrence as subsurfaces of Macfarlane manifolds.

1. INTRODUCTION

Quaternion algebras over complex number fields arise as arithmetic invariants of complete orientable finite-volume hyperbolic 3-manifolds [5, 15]. Quaternion algebras over totally real number fields are similarly associated to immersed totally-geodesic hyperbolic subsurfaces of these manifolds [15, 27]. The arithmetic properties of the quaternion algebras can be analyzed to yield geometric and topological information about the manifolds and their commensurability classes [16, 18].

In this paper we introduce an alternative geometric interpretation of these algebras, recalling that they are a generalization of the classical quaternions \mathbb{H} of Hamilton. In [20], the author elaborated on a classical idea of Macfarlane [14] to show how an involution on the complex quaternion algebra can be used to realize the action of $\text{Isom}^+(\mathfrak{H}^3)$ multiplicatively, similarly to the classical use of the standard involution on \mathbb{H} to realize the action of $\text{Isom}^+(S^2)$. Here we generalize this to a class of quaternion algebras over complex number fields and characterize them by an arithmetic condition. We define Macfarlane manifolds as those having these algebras as their invariants.

We establish the existence of arithmetic and non-arithmetic Macfarlane manifolds and in each of these classes, infinitely many non-commensurable compact and non-compact examples. We then generalize the complex quaternionic tools from [20] to develop a new algorithm for computing Dirichlet domains of Macfarlane manifolds and their immersed totally-geodesic hyperbolic subsurfaces.

Main Results. Let X be a complete orientable finite-volume hyperbolic 3-manifold. Let K and \mathcal{B} be the trace field and quaternion algebra of X , respectively.

Definition 1.1. X is *Macfarlane* if and only if

- (1) K is an imaginary quadratic extension of a real field, and
- (2) complex conjugation acts freely on the ramification set of \mathcal{B} .

The main idea (made precise by Theorem 3.3 and Corollary 4.2) can be stated informally as follows.

Theorem 1.2. *X is Macfarlane if and only if there exists an involution \dagger on \mathcal{B} which, with the quaternion norm, naturally gives rise to a 3-dimensional hyperboloid $\mathcal{M}_+^1 \subset \mathcal{B}$ over $K \cap \mathbb{R}$. Moreover, \dagger is unique and the action of $\pi_1(X)$ by orientation-preserving isometries of \mathfrak{H}^3 can be written quaternionically as*

$$\pi_1(X) \curvearrowright \mathcal{M}_+^1, \quad (\gamma, p) \mapsto \gamma p \gamma^\dagger.$$

By comparison, via Hamilton's classical result one can use the standard involution $*$ on \mathbb{H} to realize $\text{Isom}^+(S^2)$ quaternionically [20] as

$$\text{PH}^1 \curvearrowright \mathbb{H}_0^1, \quad (\gamma, p) \mapsto \gamma p \gamma^*.$$

In §4, we describe an adaptation of the main result for hyperbolic surfaces and show how, in certain instances, an immersion of a surface in a 3-manifold is sufficient for the 3-manifold to be Macfarlane. We then study other topological and arithmetic conditions under which Macfarlane manifolds arise, culminating in the following theorem.

Theorem 1.3. *There exist infinitely many non-commensurable Macfarlane manifolds in each of the following categories:*

- (1) *arithmetic and non-compact,*
- (2) *arithmetic and compact,*
- (3) *non-arithmetic and non-compact,*
- (4) *non-arithmetic and compact.*

Lastly, in §5 we use the quaternion model to provide a new algorithm for computing Dirichlet domains and illustrate the technique with some basic examples.

2. PRELIMINARIES

See [4, 21] for preliminary information on Kleinian and Fuchsian groups and hyperbolic topology. See [20] for preliminary information on algebras with involution, the standard Macfarlane space and some additional historical context. See [30] for a comprehensive treatment of quaternion algebras.

2.1. Quaternion algebras. Let K be a field with $\text{char}(K) \neq 2$. In this article K will usually be one of: \mathbb{R}, \mathbb{C} , a p -adic field, or a concrete number field i.e. $[K : \mathbb{Q}] < \infty$ with a fixed embedding $K \subset \mathbb{C}$.

Definition 2.1. Let $a, b \in K^\times$, called the *structure constants* of the algebra. The *quaternion algebra* $\left(\frac{a, b}{K}\right)$ is the associative K -algebra (with unity) $K \oplus Ki \oplus Kj \oplus Kij$, with multiplication rules $i^2 = a$, $j^2 = b$ and $ij = -ji$.

- (1) The *quaternion conjugate* of q is $q^* := w - xi - yj - zk$.
- (2) The *(reduced) norm* of q is $n(q) := qq^* = w^2 - ax^2 - by^2 + abz^2$.
- (3) The *(reduced) trace* of q is $\text{tr}(q) := q + q^* = 2w$.
- (4) q is a *pure quaternion* when $\text{tr}(q) = 0$.

We indicate conditions on the trace via subscript, for instance given a subset $E \subset \left(\frac{a, b}{K}\right)$, we write $E_0 = \{q \in E \mid \text{tr}(q) = 0\}$ and $E_+ = \{q \in E \mid \text{tr}(q) > 0\}$. We indicate conditions on the norm via superscript, for instance $E^1 = \{q \in E \mid n(q) = 1\}$.

Proposition 2.2. [30] *If $K \subset \mathbb{C}$, then \exists a faithful matrix representation of $\left(\frac{a, b}{K}\right)$ into $M_2(\mathbb{C})$. Moreover, under any such representation, n and tr correspond to the matrix determinant and trace, respectively.*

We will be interested in the K -algebra isomorphism class of $\left(\frac{a,b}{K}\right)$, which is not uniquely determined by a and b .

Theorem 2.3. [30]

- (1) Either $\left(\frac{a,b}{K}\right) \cong M_2(K)$, or $\left(\frac{a,b}{K}\right)$ is a division algebra.
- (2) $\left(\frac{a,b}{K}\right) \cong M_2(K) \iff \exists (x, y) \in K^2$ such that $ax^2 + by^2 = 1$.
- (3) For all $x \in K^\times$, $\left(\frac{a,b}{K}\right) \cong \left(\frac{b,a}{K}\right) \cong \left(\frac{ax^2,b}{K}\right)$.

It follows that for any K , there is the quaternion K -algebra $\left(\frac{1,1}{K}\right) \cong M_2(K)$. Moreover, if $\left(\frac{a,b}{K}\right)$ is not a division algebra, then its isomorphism class is unique and can be represented by $\left(\frac{1,1}{K}\right)$. Thus we now focus on quaternion division algebras.

Example 2.4. See [30] for proofs of (3) and (4) below.

- (1) Over \mathbb{R} , the only quaternion division algebra up to isomorphism is $\mathbb{H} := \left(\frac{-1,-1}{\mathbb{R}}\right)$, Hamilton's quaternions.
- (2) There are no quaternion division algebras over \mathbb{C} .
- (3) Over a p -adic field, there is a unique quaternion division algebra up to isomorphism.
- (4) Over a number field, there are infinitely many non-isomorphic quaternion division algebras.

This raises the question of how to tell, when K is a number field, whether or not two quaternion K -algebras are isomorphic. This can be done by investigating the local algebras with respect to the places of K , in the following sense.

Let K be a (concrete) number field and let $\mathcal{B} = \left(\frac{a,b}{K}\right)$. For a place v of K , let K_v be the completion of K with respect to v . To each v , we associate an embedding $\sigma : K \hookrightarrow K_v$ as described in the following paragraph, and then define the localization of \mathcal{B} with respect to v as $\mathcal{B}_v := \mathcal{B} \otimes_\sigma K_v$.

If v is infinite (i.e. Archimedean), then it corresponds (up to complex conjugation) to an element of the Galois group of K over \mathbb{Q} , under which the completion of the image of K is either \mathbb{R} or \mathbb{C} , and we define σ as the corresponding embedding. So if $\sigma(K) \subset \mathbb{R}$ then $\mathcal{B}_v = \left(\frac{\sigma(a), \sigma(b)}{\mathbb{R}}\right)$, which is isomorphic to either \mathbb{H} or $M_2(\mathbb{R})$. If $\sigma(K) \not\subset \mathbb{R}$ then $\mathcal{B}_v = \left(\frac{\sigma(a), \sigma(b)}{\mathbb{C}}\right)$, which is always isomorphic to $M_2(\mathbb{C})$. If v is finite (i.e. non-Archimedean), then it corresponds to a prime ideal $\mathfrak{P} \triangleleft \mathbb{Z}_K$, where \mathbb{Z}_K is the ring of integers of K . In this case we define σ as the identity embedding into the corresponding p -adic field, thus $\mathcal{B}_v := \left(\frac{a,b}{K_{\mathfrak{P}}}\right)$.

Definition 2.5.

- (1) \mathcal{B} is *ramified* if it is a division algebra, and is *split* if $\mathcal{B} \cong M_2(K)$.
- (2) \mathcal{B} is *ramified at v* (respectively, *split at v*) if \mathcal{B}_v is *ramified* (respectively, *split*).
- (3) $\text{Ram}(\mathcal{B})$ is the set of real embeddings and prime ideals that correspond to the places where \mathcal{B} is ramified.

The set $\text{Ram}(\mathcal{B})$ provides the desired classification of isomorphism classes of quaternion algebras over number fields.

Theorem 2.6. [30]

- (1) \mathcal{B} is split if and only if $\text{Ram}(\mathcal{B}) = \emptyset$.
- (2) $\text{Ram}(\mathcal{B})$ uniquely determines the isomorphism class of \mathcal{B} .
- (3) $\text{Ram}(\mathcal{B})$ is a finite set of even cardinality, and every such set of places of K occurs as $\text{Ram}(\mathcal{B})$ for some \mathcal{B} .

2.2. The arithmetic of hyperbolic 3-manifolds. Let X be a complete orientable finite-volume hyperbolic 3-manifold. Then $\pi_1(X) \cong \Gamma < \text{PSL}_2(\mathbb{C})$ for some discrete group Γ (i.e. Γ is a Kleinian group). Let $\hat{\Gamma} := \{ \pm \gamma \mid \{\pm \gamma\} \in \Gamma \} < \text{SL}_2(\mathbb{C})$.

Definition 2.7.

- (1) The *trace field* of Γ is $K\Gamma := \mathbb{Q}(\{\text{tr}(\gamma) \mid \gamma \in \hat{\Gamma}\})$.
- (2) The *quaternion algebra* of Γ is $B\Gamma := \{ \sum_{\ell=1}^n t_\ell \gamma_\ell \mid t_\ell \in K\Gamma, \gamma_\ell \in \hat{\Gamma}, n \in \mathbb{N} \}$.

Remark 2.8. In the literature these are usually denoted by $k_0\Gamma$ and $A_0\Gamma$, but we write them differently to avoid confusion with the notation for pure quaternions.

$K\Gamma$ is a number field and $B\Gamma$ is a quaternion algebra over $K\Gamma$ [15]. By Mostow-Prasad rigidity, these are manifold invariants in the sense that if Γ and Γ' are two faithful representations of $\pi_1(X)$, then $K\Gamma = K\Gamma'$ and $B\Gamma \cong B\Gamma'$ via a $K\Gamma$ -algebra isomorphism (though the converse does not hold). So we may also refer to them as the *trace field* and *quaternion algebra* of X up to homeomorphism.

Definition 2.9. Let $\Gamma^{(2)} := \langle \gamma^2 \mid \gamma \in \Gamma \rangle$.

- (1) The *invariant trace field* of Γ is $k\Gamma := K\Gamma^{(2)}$.
- (2) The *invariant quaternion algebra* of Γ is $A\Gamma := B\Gamma^{(2)}$.

These likewise are invariants of X , but have the stronger property of being commensurability invariants. That is, if Γ is commensurable up to conjugation to some Kleinian group Γ' , then $k\Gamma = k\Gamma'$ and $A\Gamma \cong A\Gamma'$ (though the converse does not hold) [18].

We call X *arithmetic* if Γ is an arithmetic group in the sense of [6], but this admits the following alternative characterization [15].

Definition 2.10.

- (1) Γ (or X) is *derived from a quaternion algebra* if there exists a quaternion algebra \mathcal{B} over a field K with exactly one complex place σ , such that \mathcal{B} is ramified at every real place of K , and \exists an order $\mathcal{O} \subset \mathcal{B}$ such that Γ is a finite-index subgroup of $\text{P}\sigma(\mathcal{O})^1$.
- (2) Γ (or X) is *arithmetic* if it is commensurable up to conjugation to one that is derived from a quaternion algebra.

If Γ is derived from a quaternion algebra \mathcal{B} over a field K , then $K = K\Gamma = k\Gamma$ and $\mathcal{B} = B\Gamma = A\Gamma$. If Γ is arithmetic, then $\Gamma^{(2)}$ is derived from a quaternion algebra, so then $A\Gamma^{(2)} = B\Gamma^{(2)}$. In general, $\Gamma^{(2)}$ is a finite-index subgroup of Γ . [18]

While $k\Gamma$ and $A\Gamma$ are generally more suitable to the application of arithmetic, we will work instead with $B\Gamma$ so that we may take advantage of the natural embedding $\hat{\Gamma} \hookrightarrow B\Gamma$. (To simplify notation, and where it will not cause confusion, we will often refer to an element $\{\pm \gamma\} \in \hat{\Gamma}$ by a representative $\gamma \in \Gamma$.) Often, $A\Gamma$ and $B\Gamma$ coincide (though not always [22]).

Proposition 2.11.

- (1) $k\Gamma = K\Gamma$ if and only if $A\Gamma \cong B\Gamma$.
- (2) If \mathcal{H}^3/Γ is a knot or link complement, then $A\Gamma \cong B\Gamma$.

Proof. We prove (1), and see [16] for (2). The reverse implication is immediate. For the forward implication, note that $A\Gamma \subset B\Gamma$ and both are 4-dimensional vector spaces, so if they are over the same field then they must be the same. \square

We now collect some important properties of these invariants.

Theorem 2.12. [16]

- (1) If X is non-compact, then $B\Gamma \cong \left(\frac{1,1}{K\Gamma}\right)$ and $A\Gamma \cong \left(\frac{1,1}{k\Gamma}\right)$
- (2) If X is non-compact and arithmetic, then $\exists d \in \mathbb{N}$ such that $k\Gamma = \mathbb{Q}(\sqrt{-d})$.
- (3) If X is compact and arithmetic, then $A\Gamma$ is a division algebra.

3. MACFARLANE QUATERNION ALGEBRAS AND $\text{Isom}^+(\mathfrak{H}^3)$

Our goal in this section is to show that the arithmetic characterization of Macfarlane manifolds given by Definition 1.1 admits the geometric interpretation given by Theorem 1.2. If \mathcal{B} is the quaternion algebra of a Macfarlane manifold, we will call \mathcal{B} Macfarlane as well. To define this property more generally, let \mathcal{B} be a quaternion algebra over a field $K \subset \mathbb{C}$ (not necessarily a number field).

Definition 3.1. \mathcal{B} is *Macfarlane* if

- (1) $\exists F \subset \mathbb{R}$ and $\exists d \in F^+$ such that $K = F(\sqrt{-d})$, and
- (2) complex conjugation acts freely on $\text{Ram}(\mathcal{B})$, i.e. is closed and has no fixed points.

Example 3.2.

- (1) $\left(\frac{1,1}{\mathbb{C}}\right)$ is Macfarlane because $\mathbb{C} = \mathbb{R}(\sqrt{-1})$ and $\text{Ram}\left(\frac{1,1}{\mathbb{C}}\right) = \emptyset$.
- (2) The figure-8 knot complement and its quaternion algebra $\left(\frac{1,1}{\mathbb{Q}(\sqrt{-3})}\right)$ are Macfarlane.
- (3) The quaternion algebra \mathcal{B} over $\mathbb{Q}(\sqrt{-5})$ with $\text{Ram}(\mathcal{B}) = \{(3, 1 + \sqrt{-5}), (3, 1 - \sqrt{-5})\}$ is Macfarlane, and so is any manifold derived from it.

We now give a result which is similar to and implies Theorem 1.2, but in terms of \mathcal{B} and with more detail. First notice that by Proposition 2.2, PB^1 is isomorphic to a subgroup of $\text{PSL}_2(\mathbb{C})$. This implies an injection $\text{PB}^1 \hookrightarrow \text{Isom}^+(\mathfrak{H}^3)$, but for Macfarlane quaternion algebras we can make this explicit.

Theorem 3.3. \mathcal{B} is Macfarlane if and only if it admits an involution \dagger such that $\text{Sym}(\mathcal{B}, \dagger)$ (which we denote by \mathcal{M}), equipped with the restriction of the quaternion norm, is a quadratic space of signature $(1, 3)$ over $\text{Sym}(K, \dagger)$.

Moreover, \dagger is unique and, letting $\mathcal{M}_+^1 = \{p \in \mathcal{M} \mid \text{tr}(p) > 0, \text{n}(p) = 1\}$, a faithful action of PB^1 upon \mathfrak{H}^3 by orientation-preserving isometries is defined by the group action

$$\mu_{\mathcal{B}} : \text{PB}^1 \curvearrowright \mathcal{M}_+^1, \quad (\gamma, p) \mapsto \gamma p \gamma^\dagger.$$

Definition 3.4. \mathcal{M} as in Theorem 3.3 is called a *Macfarlane space*.

Proof of Theorem 3.3. First we show that the existence of an involution as in the Theorem is equivalent to a condition on the field and structure constants of the algebra, up to isomorphism.

Lemma 3.5. \mathcal{B} admits an involution with the properties described in the Theorem if and only if $\mathcal{B} \cong \left(\frac{a,b}{F(\sqrt{-d})}\right)$ for some $F \subset \mathbb{R}$ and $a, b, d \in F^+$.

The reverse direction of this, in the case where $\mathcal{B} = \left(\frac{a,b}{F(\sqrt{-d})}\right)$, is Theorem 7.2 of [20]. This generalizes to $\mathcal{B} \cong \left(\frac{a,b}{F(\sqrt{-d})}\right)$ because an isomorphism between quaternion algebras is also a quadratic space isometry with respect to the quaternion norms [30], thus it transfers the multiplicative structure, the involution and the Macfarlane space.

So it suffices to prove the forward direction, and we do this via a series of claims. Let \mathcal{B} be a quaternion algebra over a field K and suppose \mathcal{B} admits an involution \dagger with the properties described in Theorem 3.3.

Claim 3.6. *K is of the form $F(\sqrt{-d})$ where $F = \text{Sym}(K, \dagger) \subset \mathbb{R}$ and $d \in F^+$, and $\dagger|_K$ is complex conjugation.*

Proof. If K were real, then n would be a quadratic form of signature $(2, 2)$, making it impossible for \mathcal{B} to contain a subspace of signature $(1, 3)$, thus $K \not\subseteq \mathbb{R}$. On the other hand, for a space to have nontrivial signature over $\text{Sym}(K, \dagger)$, we must have $\text{Sym}(K, \dagger) \subset \mathbb{R}$. This means \dagger is an involution of the second kind which implies $[K : \text{Sym}(K, \dagger)] = 2$ [13, 20], i.e. $K = F(\sqrt{-d})$ as in the Claim.

We now show that $\dagger|_K$ is complex conjugation. Since $-d \in F = \text{Sym}(K, \dagger)$, we have

$$(\sqrt{-d}^\dagger)^2 = (\sqrt{-d}^2)^\dagger = (-d)^\dagger = -d,$$

thus $\sqrt{-d}^\dagger = \pm\sqrt{-d}$. Since $\sqrt{-d} \notin \text{Sym}(K, \dagger)$, this leaves $\sqrt{-d}^\dagger = -\sqrt{-d}$. \square

Write $\mathcal{M} = \text{Sym}(\mathcal{B}, \dagger)$. We are going to use the signature of $n|_{\mathcal{M}}$ to prove that \mathcal{B} has real structure parameters up to isomorphism, but a priori we do not know what \mathcal{M} is. So we will first need to establish that \mathcal{M} includes enough linearly independent elements of \mathcal{B} , in the following sense.

Claim 3.7. $\text{Span}_K(\mathcal{M}) = \mathcal{B}$.

Proof. We know that $F \subset \mathcal{M}$ and $\text{Span}_K(F) = K$, so it suffices to prove $\text{Span}_K(\mathcal{M}_0) = \mathcal{B}_0$.

Let $E = \{s_1, s_2, s_3\}$ be a basis for \mathcal{M}_0 over F and assume by way of contradiction that E is not linearly independent over K . Then $\exists k_\ell \in K$ such that $\sum_{\ell=1}^3 k_\ell s_\ell = 0$. Since $K = F(\sqrt{-d})$, we have that each $k_\ell = f_{\ell,1} + f_{\ell,2}\sqrt{-d}$ for some $f_{\ell,1}, f_{\ell,2} \in F$. Substituting these into $\sum_{\ell=1}^3 k_\ell s_\ell = 0$ and rearranging terms, we get

$$f_{1,1}s_1 + f_{2,1}s_2 + f_{3,1}s_3 = -\sqrt{-d}(f_{1,2}s_1 + f_{2,2}s_2 + f_{3,2}s_3).$$

But $f_{1,1}s_1 + f_{2,1}s_2 + f_{3,1}s_3$ and $f_{1,2}s_1 + f_{2,2}s_2 + f_{3,2}s_3$ both lie in \mathcal{M} , so are fixed by \dagger , meanwhile by the previous claim, $\sqrt{-d}^\dagger = -\sqrt{-d}$. So applying \dagger to both sides of the equation gives

$$f_{1,1}s_1 + f_{2,1}s_2 + f_{3,1}s_3 = \sqrt{-d}(f_{1,2}s_1 + f_{2,2}s_2 + f_{3,2}s_3).$$

Adding the last two displayed equations then gives that $f_{1,2}s_1 + f_{2,2}s_2 + f_{3,2}s_3 = 0$. Since $f_{1,2}, f_{2,2}, f_{3,2} \in F$, this contradicts that E is a basis for \mathcal{M}_0 over F .

We conclude that E is linearly independent over K , giving that

$$\dim_K(\text{Span}_K(E)) = \dim_K(\text{Span}_K(\mathcal{M}_0)) = 3,$$

which forces $\text{Span}_K(\mathcal{M}_0) = \mathcal{B}_0$, as desired. \square

Claim 3.8. $\mathcal{B} \cong \left(\frac{a,b}{F(\sqrt{-d})}\right)$ for some $a, b \in F^+$.

Proof. The norm $n|_{\mathcal{M}}$ is a real-valued quadratic form of signature $(1, 3)$, so there exists an orthogonal basis D for \mathcal{M} so that the Gram matrix for $n|_{\mathcal{M}}$ with respect to D is a diagonal matrix $G_{n|_{\mathcal{M}}}^D$, with diagonal of the form $(f_1, -f_2, -f_3, -f_4)$ for some $f_\ell \in F^+$. Since $\text{Span}_K(\mathcal{M}) = \mathcal{B}$, this same D is also an orthogonal basis for \mathcal{B} over K .

Let C be the standard basis $\{1, i, j, ij\}$ for \mathcal{B} . Then C is another orthogonal basis for \mathcal{B} over K and, in particular, the Gram matrix G_n^C for n with respect to C is the diagonal matrix with diagonal $(1, -a, -b, ab)$.

Now while $G_{n|_{\mathcal{M}}}^D$ and G_n^C are not congruent over F , they are congruent over K because D and C are both bases for \mathcal{B} , i.e. $\exists \delta \in \text{GL}_4(K)$ such that

$$(3.9) \quad \delta G_n^D \delta^\top = G_n^C.$$

But since G_n^D and G_n^C are diagonal and nonzero on their diagonals, δ must also be diagonal and nonzero on its diagonal, i.e. $\exists x_\ell \in K^\times$ such that the diagonal of δ is (x_1, x_2, x_3, x_4) . Plugging in to (3.9) and solving for the f_ℓ gives

$$f_1 = \frac{1}{x_1^2} \quad f_2 = \frac{a}{x_2^2} \quad f_3 = \frac{b}{x_3^2} \quad f_4 = \frac{-ab}{x_4^2}.$$

Now let $\mathcal{B}' = \left(\frac{f_2, f_3}{F(\sqrt{-d})} \right)$ and recall that $f_2, f_3 \in F^+$. Then \mathcal{B} has the desired form because, by Theorem 2.3,

$$\mathcal{B}' \cong \left(\frac{f_2 x_2^2, f_3 x_3^2}{F(\sqrt{-d})} \right) = \mathcal{B}.$$

□

This completes the proof of Lemma 3.5.

Now to complete the proof of Theorem 3.3, we show that the condition on the isomorphism class of the symbol $\left(\frac{a, b}{K} \right)$ from Lemma 3.5 is equivalent to the arithmetic characterization of Macfarlane quaternion algebras given by Definition 3.1.

Lemma 3.10. *Let \mathcal{B} be a quaternion algebra over $K = F(\sqrt{-d})$ where $F \subset \mathbb{R}$ and $d \in F^+$. Complex conjugation acts freely on $\text{Ram}(\mathcal{B})$ if and only if $\exists a, b \in F^+$ such that $\mathcal{B} \cong \left(\frac{a, b}{K} \right)$.*

Proof. With a, b, F and K as in the statement, notice that $\left(\frac{a, b}{K} \right) = \left(\frac{a, b}{F} \right) \otimes_F K$. Also if a (or b) is negative, then by Theorem 2.3, we can replace it by $-ad$ (or $-bd$) without changing the isomorphism class. So it suffices to prove that the condition on $\text{Ram}(\mathcal{B})$ is equivalent to the existence of a quaternion algebra \mathcal{A} over F such that $\mathcal{B} \cong \mathcal{A} \otimes_F K$.

Since K is an imaginary quadratic extension of a real field, it has no real places, making $\text{Ram}(\mathcal{B})$ a set of prime ideals of \mathbb{Z}_K . Denote an arbitrary prime ideal of \mathbb{Z}_K by \mathfrak{P} . So for complex conjugation to act freely on $\text{Ram}(\mathcal{B})$ is to say that $\forall \mathfrak{P} \in \text{Ram}(\mathcal{B})$, we have $\mathfrak{P} \neq \overline{\mathfrak{P}} \in \text{Ram}(\mathcal{B})$.

For every prime ideal $\mathfrak{P} \triangleleft \mathbb{Z}_K$, there exists a prime ideal $\mathfrak{p} \triangleleft \mathbb{Z}_F$ such that $\mathfrak{P} | \mathfrak{p}$ where $\mathfrak{p} \mathbb{Z}_K$ is one of \mathfrak{P} , \mathfrak{P}^2 , or $\mathfrak{P} \overline{\mathfrak{P}}$ where $\mathfrak{P} \neq \overline{\mathfrak{P}}$ (i.e. \mathfrak{p} is inert, ramified or split in the extension $K : F$, respectively). In the first two cases $\overline{\mathfrak{P}} = \mathfrak{P}$, and the third possibility is the only way that these prime ideals occur in complex conjugate pairs. So to say that complex conjugation acts freely on $\text{Ram}(\mathcal{B})$ is equivalent to saying that there exist prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n \triangleleft \mathbb{Z}_F$ such that

$$(3.11) \quad \text{Ram}(\mathcal{B}) = \{ \mathfrak{P}_\ell, \overline{\mathfrak{P}_\ell} \mid \mathfrak{p}_\ell \mathbb{Z}_K = \mathfrak{P}_\ell \overline{\mathfrak{P}_\ell}, \mathfrak{P}_\ell \neq \overline{\mathfrak{P}_\ell}, \ell = 1, \dots, n \}.$$

Our task is thus to prove that $\text{Ram}(\mathcal{B})$ has this form if and only if \exists a quaternion algebra \mathcal{A} over F such that $\mathcal{B} \cong \mathcal{A} \otimes_F K$.

For the reverse direction, suppose that $\mathcal{B} \cong \mathcal{A} \otimes_F K$ as above. As already noted, any infinite place of \mathcal{A} becomes complex in \mathcal{B} , so does not contribute to $\text{Ram}(\mathcal{B})$. So let $\mathfrak{p} \triangleleft \mathbb{Z}_F$ be a prime ideal and we investigate its contribution to $\text{Ram}(\mathcal{B})$. If $\mathfrak{p} \notin \text{Ram}(\mathcal{A})$, i.e. $\mathcal{A}_{\mathfrak{p}}$ is split, then \mathcal{B} is split at the primes lying above \mathfrak{p} , meaning that those do not occur in $\text{Ram}(\mathcal{B})$. So we can assume $\mathfrak{p} \in \text{Ram}(\mathcal{A})$.

If $\mathfrak{p}\mathbb{Z}_K$ has the form \mathfrak{P} or \mathfrak{P}^2 , then $K_{\mathfrak{P}} \cong F_{\mathfrak{p}}(x)/(x^2 + d)$, a quadratic extension of $F_{\mathfrak{p}}$. Since $F_{\mathfrak{p}}$ is a p -adic field, $\mathcal{A}_{\mathfrak{p}}$ contains all of its quadratic extensions [30]. Then $K_{\mathfrak{P}} \hookrightarrow \mathcal{A}_{\mathfrak{p}}$, which implies that $\mathcal{A}_{\mathfrak{p}} \otimes_{F_{\mathfrak{p}}} K_{\mathfrak{P}}$ is split, but this equals $\mathcal{B}_{\mathfrak{P}}$. Thus $\mathfrak{P} \notin \text{Ram}(\mathcal{B})$.

If $\mathfrak{p}\mathbb{Z}_K$ has the form $\mathfrak{P}\overline{\mathfrak{P}}$, then $K_{\mathfrak{P}} = F_{\mathfrak{p}}$, and then $\mathcal{A}_{\mathfrak{p}} \cong \mathcal{B}_{\mathfrak{P}}$ (and likewise for $\overline{\mathfrak{P}}$). So the primes above \mathfrak{p} occur in $\text{Ram}(\mathcal{B})$, giving it the desired form.

For the forward implication, suppose $\exists \mathfrak{p}_1, \dots, \mathfrak{p}_n \triangleleft \mathbb{Z}_F$ such that Equation (3.11) holds. From Theorem 2.3 we know that for any finite even set of places of F , there exists a quaternion algebra over F having that as its ramification set. If n is even, let \mathcal{A} be the quaternion algebra over F with $\text{Ram}(\mathcal{A}) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. If n is odd, choose any one of the (infinitely many) prime ideals of $\mathfrak{p} \triangleleft \mathbb{Z}_F$ that does not split in the extension to K , and let \mathcal{A} satisfy $\text{Ram}(\mathcal{A}) = \{\mathfrak{p}, \mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. The above argument then gives that $\text{Ram}(\mathcal{A} \otimes_F K) = \text{Ram}(\mathcal{B})$, thus $\mathcal{B} \cong \mathcal{A} \otimes_F K$. \square

This completes the proof of Theorem 3.3. The following consequence has computational advantages which will be exploited in §5.

Corollary 3.12. *If \mathcal{B} is Macfarlane, then there is an isomorphism $\mathcal{B} \cong \left(\frac{a,b}{F(\sqrt{-d})} \right)$ where $a, b, d \in F^+$. In this case the Macfarlane space is*

$$\mathcal{M} = F \oplus Fi \oplus Fj \oplus \sqrt{-d}Fij$$

and for $q = w + xi + yj + zij \in \mathcal{B}$ with $w, x, y, z \in F(\sqrt{-d})$, the involution \dagger is given by

$$(3.13) \quad q^{\dagger} = \overline{w} + \overline{x}i + \overline{y}j - \overline{z}ij.$$

A final remark on Theorem 3.3 is that even though we are using \mathcal{M}_+^1 as a hyperboloid model for the group action of interest, it is technically not a model for \mathfrak{H}^3 unless $F = \mathbb{R}$. We will study the case where F is a concrete number field embedded in \mathbb{R} . If a complete model for \mathfrak{H}^3 is desired, one is given by $(\mathcal{M} \otimes_F \mathbb{R})_+^1$ but we will generally not need this.

4. MACFARLANE MANIFOLDS

In this section we explore the various conditions in which Macfarlane manifolds arise. First we clarify why Theorem 1.2 about Macfarlane manifolds follows from Theorem 3.3 about Macfarlane quaternion algebras. Then we look at an adaptation of our results to hyperbolic surfaces, and see how immersed subsurfaces sometimes give rise to Macfarlane 3-manifolds. The remainder of the section culminates in a proof of Theorem 1.3.

Let X denote a complete orientable finite-volume hyperbolic 3-manifold with Kleinian group $\Gamma \cong \pi_1(X)$. Let $K = K\Gamma$ and $\mathcal{B} = B\Gamma$.

Definition 4.1. When X is Macfarlane and $\mathcal{M} \subset \mathcal{B}$ is its Macfarlane space as in Theorem 3.3, define $\mathcal{I}_{\Gamma} := \mathcal{M}_+^1$ and call this a *quaternion hyperboloid model* for Γ (or X).

It is immediate that \mathcal{I}_{Γ} , up to quadratic space isometry over $K\Gamma$, is a manifold invariant.

By the definition of $B\Gamma = \mathcal{B}$, there is no confusion in speaking of Γ quaternionically, as lying in \mathcal{PB}^1 rather than in $\mathrm{PSL}_2(\mathbb{C})$. In this way, Γ (up to choice of representatives in $\hat{\Gamma}$) and \mathcal{I}_Γ are both subsets of \mathcal{B} , making sense of the following, which gives Theorem 1.2.

Corollary 4.2. *If X is Macfarlane, then the action of Γ by orientation-preserving isometries of \mathfrak{H}^3 is faithfully represented by*

$$\mu_\Gamma : \Gamma \hookrightarrow \mathcal{I}_\Gamma, \quad (\gamma, p) \mapsto \gamma p \gamma^\dagger.$$

4.1. Hyperbolic Surfaces and Subsurfaces. The author showed in [20] how the representation of $\mathrm{Isom}^+(\mathfrak{H}^3)$ in $\left(\frac{1,1}{\mathbb{C}}\right)$ restricts to a representation of $\mathrm{Isom}^+(\mathfrak{H}^2)$ in $\left(\frac{1,1}{\mathbb{R}}\right)$. Similarly, we seek an analogue of Theorem 1.2 for hyperbolic subsurfaces of 3-manifolds. This is possible to some extent but we must proceed with care because there are important differences between the 3-dimensional and 2-dimensional settings.

Let S denote a complete orientable finite-volume hyperbolic surface. The group $\pi_1(S)$ admits discrete faithful representations into $\mathrm{PSL}_2(\mathbb{R})$ but in the absence of Mostow-Prasad rigidity, an isomorphism class of such representations only gives S up to isometry, not homeomorphism. Furthermore, a Fuchsian group can contain transcendental traces, meaning its trace field need not be a number field. We will soon see a way of getting around this but for now let us think of S up to isometry.

Let $\Delta < \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian group. The *(invariant) trace field* of Δ and *(invariant) quaternion algebra* of Δ are defined in the same way as in Definitions 2.7 and 2.9, and we denote them similarly by $(k\Delta)$ $K\Delta$ and $(A\Delta)$ $B\Delta$, respectively. These have properties similar to what we saw in the Kleinian setting. For instance, $B\Delta$ and $A\Delta$ are quaternion algebras over $K\Delta$ and $k\Delta$ respectively [25], and $A\Delta$ is a commensurability invariant [28].

The results from §6 of [20] along with the proof of Lemma 3.5 then give the corollary below, after the following observations. The field $K\Delta$ is real, and so now the involution \dagger is of the first kind. That is, $\mathrm{Sym}(K\Delta, \dagger) = K\Delta$ and $\mathrm{Sym}(B\Delta, \dagger)$ is comprised of $K\Delta$ and the unique 2-dimensional negative-definite subspace with respect to the norm on $B\Delta$.

Corollary 4.3. *If $B\Delta \cong \left(\frac{a,b}{K\Delta}\right)$ for some $a, b > 0$, then it admits an involution \dagger such that $\mathrm{Sym}(B\Delta, \dagger)$ (which we denote by \mathcal{L}), equipped with the restriction of the quaternion norm, is a quadratic space of signature $(1, 2)$ over $K\Delta$.*

Moreover, \dagger is unique and, letting $\mathcal{L}_+^1 = \{p \in \mathcal{L} \mid \mathrm{tr}(p) > 0, n(p) = 1\}$, a faithful action of Δ upon \mathfrak{H}^2 by orientation-preserving isometries is defined by the group action

$$\mu_\Delta : \Delta \hookrightarrow \mathcal{L}_+^1, \quad (\gamma, p) \mapsto \gamma p \gamma^\dagger.$$

Definition 4.4. We call the space $\mathcal{L} \subset B\Delta$ as above a *restricted Macfarlane space*, and we call the space $\mathcal{I}_\Delta := \mathcal{L}_+^1$ a *quaternion hyperboloid model* for Δ .

We next give a way of realizing Macfarlane 3-manifolds using Fuchsian groups. Despite the absence of rigidity, in the homeomorphism class of S there always exists a representation $\Delta < \mathrm{PSL}_2(\mathbb{R})$ of $\pi_1(S)$ such that $K\Delta$ is a number field [26]. When S is an immersed closed totally-geodesic subsurface of a hyperbolic 3-manifold X , there exists an injection $\pi_1(S) \hookrightarrow \pi_1(X)$, which implies such a Δ [16]. In this context, we think of S as the isometry class of surfaces induced by the homeomorphisms of X .

Proposition 4.5. *If $K = F(\sqrt{-d})$ for some $F \subset \mathbb{R}$ and $d \in F^+$, and X contains an immersed closed totally-geodesic surface, then X is Macfarlane.*

Proof. First let $S \subset X$ be a surface as in (1). Then $\pi_1(S)$ has a Fuchsian representation $\Delta < \mathrm{PSL}_2(\mathbb{R})$ and $\pi_1(X)$ has a Kleinian representation $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ such that $\Delta < \Gamma$. Then $K\Delta \subset K \cap \mathbb{R} = F$. Therefore $B\Delta \subset B\Gamma$ is a quaternion subalgebra over a subfield of F . Hence $\exists a, b \in F$ so that $B\Delta = \left(\frac{a,b}{K\Delta}\right)$. But then $B\Delta \otimes_{K\Delta} K = \left(\frac{a,b}{F(\sqrt{-d})}\right) \subset B\Gamma$. So by Proposition 2.11, $B\Gamma = \left(\frac{a,b}{F(\sqrt{-d})}\right)$, and then by Lemma 3.10, X is Macfarlane. \square

Remark 4.6. When an immersion occurs as above, the action μ_Γ as given in Theorem 1.2 restricts to the action μ_Δ as given in Corollary 4.3. An example of this will be studied in §5.

4.2. Arithmetic Macfarlane Manifolds. For X to be arithmetic and Macfarlane, its invariant trace field $k\Gamma$ must be quadratic. To see this, recall that the trace field of a Macfarlane manifold is $F(\sqrt{-d})$ for some $F \subset \mathbb{R}$ and $d \in K^+$, and that the invariant trace field is a complex subfield of this, forcing it to be of the form $F'(\sqrt{-d'})$ with $F' \subset F$ and $d' \in F'^+$. By Definition 2.10, the invariant trace field of an arithmetic Kleinian group can have only one complex place, which forces $F' = \mathbb{Q}$.

Quadratic fields do not have any real embeddings, therefore an order of a quaternion algebra over one of these fields will always give rise to a Kleinian group derived from a quaternion algebra. We use this to construct examples of arithmetic Macfarlane manifolds and prove parts (1) and (2) of Theorem 1.3.

4.2.1. Non-compact Arithmetic Macfarlane Manifolds. Every arithmetic non-compact Γ is commensurable to a Bianchi group [16], which is a group of the form $\mathrm{PSL}_2(\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})})$ where $d \in \mathbb{N}$. Such a group is derived from the quaternion algebra $\left(\frac{1,1}{\mathbb{Q}(\sqrt{-d})}\right)$. It follows that whenever X is non-compact and arithmetic, $A\Gamma$ will be of the form $\left(\frac{1,1}{\mathbb{Q}(\sqrt{-d})}\right)$, so that if $K\Gamma = k\Gamma$, then X will be Macfarlane. (While this is often the case, it is not always [22]. For example the manifold m009 in the cusped census has invariant trace field $\mathbb{Q}(\sqrt{-7})$ but trace field $\mathbb{Q}\left(\sqrt{\frac{5-\sqrt{-7}}{2}}\right)$.)

Lemma 4.7. *There are infinitely many commensurability classes of non-compact arithmetic Macfarlane manifolds.*

Proof. For each square-free $d \in \mathbb{N}$, take a torsion-free subgroup $\Gamma < \mathrm{PSL}_2(\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})})$ (whose existence is guaranteed by Selberg's Lemma). If $K\Gamma$ is not of the form $\mathbb{Q}(\sqrt{-d})$, then form $\Gamma^{(2)}$. Since Γ is arithmetic, $\Gamma^{(2)}$ is derived from $\left(\frac{1,1}{\mathbb{Q}(\sqrt{-d})}\right)$, so that $B\Gamma^{(2)} = \left(\frac{1,1}{\mathbb{Q}(\sqrt{-d})}\right)$, which is Macfarlane by Lemma 3.5. Since $A\Gamma^{(2)} = B\Gamma^{(2)}$ each choice of d gives a group in a different commensurability class. \square

Example 4.8. Arithmetic link complements are Macfarlane, by Proposition 2.11.

- (1) The figure-8 knot complement is Macfarlane, with trace field $\mathbb{Q}(\sqrt{-3})$.
- (2) The Whitehead link is Macfarlane, with trace field $\mathbb{Q}(\sqrt{-1})$.
- (3) The six-component chain link is Macfarlane, with trace field $\mathbb{Q}(\sqrt{-15})$.

4.2.2. Compact Arithmetic Macfarlane Manifolds. To construct these we again work over the quadratic fields. While there is only one split quaternion algebra over each field, there are infinitely many ramified ones. We use this to get something stronger than just a direct analog of Lemma 4.7.

Lemma 4.9. *For every square-free $d \in \mathbb{N}$, there are infinitely many commensurability classes of compact arithmetic Macfarlane manifolds having the trace field $\mathbb{Q}(\sqrt{-d})$.*

Proof. First observe that for each choice of square-free $d \in \mathbb{N}$, there are infinitely many non-similar quadratic forms of the form $-ax^2 - by^2 + abz^2 \in \mathbb{Q}(\sqrt{-d})[x, y, z]$ with $a, b \in \mathbb{Q}^+$. Such a quadratic form is the quaternion norm restricted to the pure quaternions in $\left(\frac{a, b}{\mathbb{Q}(\sqrt{-d})}\right)$, so that for each pair of non-similar quadratic forms there is a pair of non-isomorphic quaternion algebras [30]. Since only one of these isomorphism classes is split, there are infinitely many that are ramified. Let $\mathcal{B} = \left(\frac{a, b}{\mathbb{Q}(\sqrt{-d})}\right)$ be one of these and let $\Gamma' = \text{PO}^1$ for some order $\mathcal{O} \subset \mathcal{B}$.

Since $\mathbb{Q}(\sqrt{-d})$ has a unique complex place and no real places, Γ' is a Kleinian group derived from \mathcal{B} , therefore $A\Gamma' = B\Gamma' = \mathcal{B}$. So \mathcal{B} determines the commensurability class of Γ' , and since $a, b, d \in \mathbb{Q}^+$, \mathcal{B} is Macfarlane. Then since Γ' is arithmetic and \mathcal{B} is ramified, Γ' is cocompact.

If Γ' has torsion, resolve this similarly to in the proof of Lemma 4.7: take a finite-index torsion-free subgroup $\Gamma < \Gamma'$ (this necessarily satisfies $A\Gamma = A\Gamma'$) and if $B\Gamma \neq A\Gamma$, then form $\Gamma^{(2)}$. This is necessarily torsion-free, derived from \mathcal{B} , and satisfies $B\Gamma^{(2)} = \mathcal{B}$. \square

4.3. Non-arithmetic Macfarlane manifolds. We now complete the proof of Theorem 1.3 by providing an abundance of commensurability classes of non-arithmetic examples.

Lemma 4.10. *There are infinitely many commensurability classes of non-compact non-arithmetic Macfarlane manifolds.*

Proof. Recall that the quaternion algebra of a non-compact manifold is necessarily split (Theorem 2.12), so it suffices to satisfy the condition on the trace field.

In [7], an infinite class of non-commensurable link complements are generated, all having invariant trace field $\mathbb{Q}(\sqrt{-1}, \sqrt{2})$, by gluing along totally-geodesic 4-punctured spheres. Since this field is not of the form $\mathbb{Q}(\sqrt{-d})$ with $d \in \mathbb{N}$ and these manifolds are non-compact, they are non-arithmetic. Since they are link complements, by Proposition 2.11 their trace fields are also $\mathbb{Q}(\sqrt{-1}, \sqrt{2})$, which is of the desired form. \square

To realize non-arithmetic compact Macfarlane manifolds, we start some with arithmetic compact Macfarlane manifold with immersed totally-geodesic subsurfaces. Then we apply the technique of interbreeding introduced by Gromov and Piatetski-Shapiro [11]. This entails gluing together a pair of non-commensurable arithmetic manifolds along a pair of totally-geodesic and isometric subsurfaces, resulting in a non-arithmetic manifold. We use a variation on this technique introduced by Agol [1] called inbreeding, whereby one glues together a pair of geodesic subsurfaces bounding non-commensurable submanifolds of the same arithmetic manifold.

Lemma 4.11. *For every arithmetic Macfarlane manifold X containing an immersed closed totally-geodesic surface, there exist infinitely many commensurability classes of non-arithmetic Macfarlane manifolds with the same quaternion algebra as X .*

Proof. Let X be as in the statement and $\Gamma \cong \pi_1(X)$ a Kleinian group. Then X contains infinitely many non-commensurable immersed closed totally-geodesic arithmetic hyperbolic subsurfaces [16](§9.5). Let S_1 and S_2 be two of these subsurfaces. Since they are arithmetic, they each correspond to a lattice in a quaternion subalgebra over a real subfield of F , which we then deform so that S_1 and S_2 can be glued together via the identification map $f : S_1 \rightarrow S_2$ as in [1]. Then $X_{1,2} := X \star_f X$ is non-arithmetic, and the reflection involution through the identified subsurface lies in the commensurator of Γ . So with $\pi_1(X_{1,2}) \cong \Gamma_{1,2} < \text{PSL}_2(\mathbb{C})$, we have $K\Gamma = K\Gamma_{1,2}$ and $B\Gamma \cong B\Gamma_{1,2}$.

By the results in [1], there exists an infinite sequence of choices for pairs of surfaces S_ℓ, S_m as above so that the injectivity radius of $X_{\ell,m}$ gets arbitrarily small. But by Margulis' Lemma, there is a lower bound to the injectivity radius of any class of commensurable non-arithmetic manifolds. Thus in our sequence $\{X_{\ell,m}\}$, we enter a new commensurability class infinitely often as this radius approaches 0. \square

5. QUATERNION DIRICHLET DOMAINS

In this section, we apply the quaternion perspective to give a new method for computing Dirichlet domains of Macfarlane manifolds. In comparison to other methods of computing fundamental domains of hyperbolic 3-manifolds and surfaces, it is notable that the class of Macfarlane manifolds includes the range of examples detailed in the previous section. Early algorithms [12, 3, 24, 23] were specific to the non-compact arithmetic case. More recent ones have required either arithmeticity (as in [8, 29, 19]) or compactness (such as those implied by [9] and [17]).

Definition 5.1. The *Dirichlet domain* for a group Γ acting on a metric space \mathcal{X} , centered at $c \in \mathcal{X}$, is defined as

$$\mathcal{D}_\Gamma(c) := \left\{ p \in \mathcal{X} \mid \forall \gamma \in \Gamma \setminus \{1\} : d(c, p) \leq d(c, \gamma(p)) \right\}.$$

When \mathcal{X} is a geometry (for instance \mathcal{H}^3 or $(\mathcal{M} \otimes \mathbb{R})_+^1$), this is the intersection of all the half-spaces containing c formed by perpendicular bisectors of geodesics from c to the points in the orbit

$$\text{Orb}_\Gamma(c) := \{\gamma(c) \mid \gamma \in \Gamma\}.$$

Choosing c so that $\text{Stab}_\Gamma(c) = \{1\}$, we get that $\mathcal{D}_\Gamma(c)$ is a fundamental domain for \mathcal{X}/Γ . If $\{\gamma_\ell\}_\ell \subset \Gamma$ is the set such that the sides of $\mathcal{D}_\Gamma(c)$ are portions of perpendicular bisectors of geodesics from c to the points $\gamma_\ell(c)$, then the side-pairing identifications are given by applying γ_ℓ^{-1} to the side contributed by γ_ℓ , for each γ_ℓ . [4]

As before, let X be a complete orientable finite-volume hyperbolic 3-manifold and $\Gamma \cong \pi_1(X)$ a Kleinian group. Only elliptic elements of $\text{Isom}^+(\mathfrak{H}^3)$ have fixed points inside \mathfrak{H}^3 , thus $\text{Stab}_\Gamma(c) = \{1\}$ for any c in any model of \mathfrak{H}^3 , and so $\mathcal{D}_\Gamma(c)$ is always a fundamental domain for Γ . The idea now is to understand X by studying $\mathcal{D}_\Gamma(c)$ equipped with side-pairing maps on its boundary.

Since $\text{vol}(X) < \infty$, the group Γ is finitely-generated and so any Dirichlet domain for Γ has finitely many sides, which are geodesic surfaces in \mathfrak{H}^3 [4]. Provided one has a way of systematically checking whether each point in $\text{Orb}_\Gamma(c)$ contributes a side to $\mathcal{D}_\Gamma(c)$, and a way of telling when all of the sides have been found, one has an algorithm to compute $\mathcal{D}_\Gamma(c)$. In this sense there always exists such an algorithm, but not necessarily one that is useful in practice.

The current technique offers the following advantages. The Macfarlane model identifies the points in hyperbolic space with quaternions, giving them additional algebraic properties. The points in $\text{Orb}_\Gamma(c)$ have traces and pure quaternion parts which can be used to systematically order them and keep track of side-pairings. The location of points can often be translated into systems of Diophantine equations, to which the classical quaternion arithmetic gives efficient solutions. Moreover, sometimes elements of Γ are points on the hyperboloid \mathcal{I}_Γ and we can use these to locate sides of $\mathcal{D}_\Gamma(c)$ more quickly.

Let X be Macfarlane. Choose $\Gamma \cong \pi_1(X)$ so that $\mathcal{B} := B\Gamma = \left(\frac{a,b}{F(\sqrt{-d})} \right)$, $F \subset \mathbb{R}$ and $a, b, d \in F^+$ (guaranteed by Corollary 3.12), then identify Γ with its natural image in PB^1 .

Let \dagger be the involution on \mathcal{B} so that $\mathcal{M} = \text{Sym}(\mathcal{B}, \dagger)$ is the Macfarlane space. Then $\mathcal{I}_\Gamma = \mathcal{M}_+^1$ is a hyperboloid model for X . Recall the group action $\mu_\Gamma : \Gamma \times \mathcal{I}_\Gamma, (\gamma, p) \mapsto \gamma p \gamma^\dagger$ defined in Corollary 4.2, and for $(\gamma, p) \in \Gamma \times \mathcal{I}_\Gamma$, let us abbreviate $\gamma(p) := \mu_\Gamma(\gamma, p)$. Since μ_Γ acts on \mathcal{I}_Γ , we can form a Dirichlet domain for Γ in \mathcal{I}_Γ and, since $1 \in \mathcal{I}_\Gamma$ is the point lying at the bottom of the hyperboloid, $\mathcal{D}_\Gamma(1) \subset \mathcal{I}_\Gamma$ is the natural choice for this.

5.1. Orbit Points. Denote $\text{Orb}_\Gamma(1) \subset \mathcal{D}_\Gamma(1)$ by O_Γ . Applying μ_Γ to 1, we have

$$(5.2) \quad O_\Gamma = \{\gamma \gamma^\dagger \mid \gamma \in \Gamma\}.$$

Since $\forall p \in O_\Gamma, \text{tr}(p) \in F$, we use the trace to order the elements of O_Γ .

For $t \in F$, the set

$$(5.3) \quad V_t := \{p \in O_\Gamma \mid \text{tr}(p) = t\}$$

has the following structure. Take the cross-section of \mathcal{I}^Γ by the hyper-plane in \mathcal{M} which is parallel to \mathcal{M}_0 and passes through the point $t/2 \in F \subset \mathcal{M}$ (we divide t by 2 because the trace of a quaternion is twice its scalar part). This yields an ellipsoid, and V_t is the set of points from O_Γ lying on it. That is,

$$V_t = O_\Gamma \cap (\{t/2\} \oplus \mathcal{M}_0)$$

and the points of O_Γ can be partitioned into such sets. Figure 1 illustrates a 2-dimensional version of this.

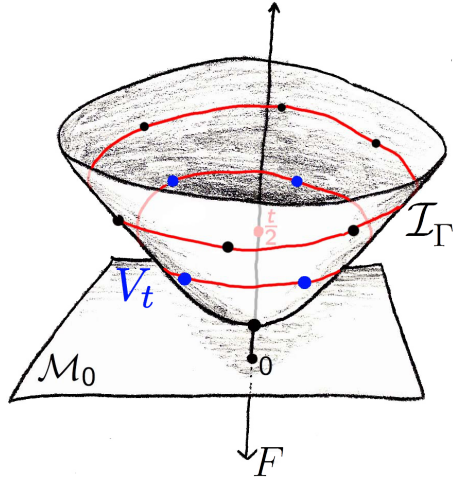


FIGURE 1. Orbit points by trace.

These sets collect the orbit points in a convenient way because they satisfy the following properties.

Proposition 5.4.

- (1) For all $t < 2$, $V_t = \emptyset$.
- (2) $V_2 = \{1\}$.
- (3) For all $t \in F$, $|V_t| < \infty$.
- (4) The set $\{t \in F \mid V_t \neq \emptyset\}$ is discrete.

Proof. The hyperboloid \mathcal{I}_Γ intersects the F -axis at 1, and stretches upward from there (with increasing trace) in every direction. Since $V_t \subset \mathcal{I}_\Gamma \cap (\{t/2\} \oplus \mathcal{M}_0)$, this proves (1), and since $\text{tr}(1) = 2$, this also proves (2).

O_Γ is discrete and V_t is contained in an ellipsoid on \mathcal{I}_Γ , thus V_t is a discrete subset of a compact set, proving (3). To prove (4), suppose there is some sequence of nonempty V_{t_ℓ} so that $\{t_\ell\}$ accumulates to t . Let $C = \{p \in \mathcal{I}_\Gamma \mid \text{tr}(p) \leq \frac{t}{2} + 1\}$. Then C is compact and contains infinitely many points from $\bigcup_\ell V_{t_\ell}$, thus these points converge to a limit point in C contradicting the discreteness of O_Γ . \square

The points $p \in V_t$ can also be described explicitly by putting the equation $n(p) = 1$ into the standard form of an ellipse.

$$(5.5) \quad V_t = \left\{ \frac{t}{2} + xi + yj + z\sqrt{-d}ij \in O_\Gamma \mid 4ax^2 + 4by^2 + 4abdz^2 = t^2 - 4 \right\}$$

is one way of doing this, though depending on what Γ is we may choose to parametrize the points differently (as in the examples at the end of this section). In the event that Γ consists of points in a quaternion order or, more generally, points with coefficients in a ring of integers, expressions such as this allow us to find orbit points by solving Diophantine equations.

Elements in the group Γ can also occur as points on the hyperboloid \mathcal{I}_Γ . These points have the following properties, which can be used to speed up the process of computing $\mathcal{D}_\Gamma(1)$.

Proposition 5.6. *If $q \in \Gamma \cap \mathcal{I}_\Gamma$ but $q \notin O_\Gamma$, then*

- (1) $q^2 \in O_\Gamma$, and
- (2) q is the midpoint between 1 and $q(1)$.

Proof. If $q \in \Gamma \cap \mathcal{I}_\Gamma$, then $q^\dagger = q$, so then $q(1) = qq^\dagger = q^2 \in O_\Gamma$, proving (1).

To prove (2), let δ be the hyperbolic translation along the geodesic that passes through 1 and q , such that $\delta(1) = q$. By Theorem 4.3 of [20], q is a hyperbolic translation along the same geodesic, so it commutes with δ . Then $\delta(q) = \delta q \delta^\dagger = \delta \delta^\dagger q = \delta(1)q = q^2$. Therefore $d(1, q)$ and $d(q, q^2)$ both equal the translation length of δ . \square

5.2. Geodesics through the Center. In constructing $\mathcal{D}_\Gamma(1)$, we will work with geodesic rays emanating from 1 and passing through the points in O_Γ . These admit a natural correspondence with equivalence classes of pure quaternions. For each point $p \in O_\Gamma$, the geodesic ray that starts at 1 and passes through p has a pure quaternion part which is a Euclidean ray. So we can identify these geodesics as follows.

Definition 5.7. The *slope* of $w + xi + yj + \sqrt{-d}zij \in O_\Gamma$, is $[x, y, z] \in F^3/F^+$.

Where possible we will indicate the slope by the minimal representative of the equivalence class where the coordinates are in \mathbb{Z}_F . This will be useful in finding $\mathcal{D}_\Gamma(1)$ and in computing side-pairings, as will be used in the examples at the end of this section.

5.3. Comparison with the Upper Half-Space Model. Since much of the existing data on the groups of interest is given in matrix form, it will be useful to be able to transfer fluidly between the current model and the more conventional one. To attain this, first apply the natural isomorphism

$$\mathcal{B} \otimes_F \mathbb{R} \rightarrow \left(\frac{1, 1}{\mathbb{C}} \right) : \quad w + xi + yj + zij \mapsto w + x\sqrt{a}i' + y\sqrt{b}j' + z\sqrt{ab}i'j'$$

where $i'^2 = j'^2 = 1$ and $i'j' = -j'i'$, which takes $(\mathcal{M} \otimes \mathbb{R})_+^1$ to a standard hyperboloid model. Then apply Theorem 5.2 of [20], as follows (where we use the same notation).

The map

$$(5.8) \quad \rho_{\mathcal{B}} : \mathcal{B} \hookrightarrow M_2(K(\sqrt{a}, \sqrt{b})), \quad w + xi + yj + zij \mapsto \begin{pmatrix} w - x\sqrt{a} & y\sqrt{b} - z\sqrt{ab} \\ y\sqrt{b} + z\sqrt{ab} & w + x\sqrt{a} \end{pmatrix}$$

is an injective $F(\sqrt{-d})$ -algebra homomorphism, and let us also write $\rho_{\mathcal{B}}^{-1}$ to mean the inverse of the corestriction $\rho|_{\rho(\mathcal{B})} : \mathcal{B} \xrightarrow{\cong} \rho_{\mathcal{B}}(\mathcal{B}) : q \mapsto \rho_{\mathcal{B}}(q)$. Write the upper half-space model as the subspace $\mathcal{H}^3 = \mathbb{R} \oplus \mathbb{R}I \oplus \mathbb{R}^+J$ of Hamilton's quaternions, where $I^2 = J^2 = -1$ and $IJ = -JI$. Then

$$(5.9) \quad \iota_{\Gamma} : \mathcal{I}_{\Gamma} \rightarrow \mathcal{H}^3, \quad w + xi + yj + \sqrt{-d}zij \mapsto \frac{y\sqrt{b} + z\sqrt{abd}I + J}{w + x\sqrt{a}}$$

is an isometry such that the Möbius action $\Gamma \times \mathcal{H}^3 \rightarrow \mathcal{H}^3$ is equal to $\iota(\mu_{\Gamma}(\rho_{\mathcal{A}}^{-1}(\cdot), \iota^{-1}(\cdot)))$. In other words, our quaternion representation μ_{Γ} of the group action from Theorem 1.2 transfers to the usual Möbius action via ι and $\rho_{\mathcal{B}}$. [20]

This gives a simple way of writing, given some $\gamma \in \Gamma$ as a matrix, the value of t in $\gamma(1) \in V_t$.

Proposition 5.10. *If $\gamma = \begin{pmatrix} r & s \\ u & v \end{pmatrix} \in \text{PSL}_2(\mathbb{C})$, then $\rho_{\mathcal{B}}^{-1}(\gamma)(1) \in V_t$ where*

$$t = |r|^2 + |s|^2 + |u|^2 + |v|^2.$$

Proof. With γ as in the statement,

$$\rho_{\mathcal{B}}(\rho_{\mathcal{B}}^{-1}(\gamma)(1)) = \gamma\gamma^{\dagger} = \begin{pmatrix} r & s \\ u & v \end{pmatrix} \begin{pmatrix} \bar{r} & \bar{u} \\ \bar{s} & \bar{v} \end{pmatrix} = \begin{pmatrix} |r|^2 + |s|^2 & r\bar{u} + s\bar{v} \\ u\bar{r} + v\bar{s} & |u|^2 + |v|^2 \end{pmatrix}.$$

Then $\text{tr}(\rho_{\mathcal{B}}^{-1}(\gamma)(1)) = \text{tr}(\gamma\gamma^{\dagger}) = |r|^2 + |s|^2 + |u|^2 + |v|^2$. □

Remark 5.11. The formula for t is the square of the Frobenius norm of γ .

5.4. The Algorithm. The technique outlined here is illustrated in the following two subsections, where we carry it out on some basic examples. To discuss the algorithm in general, we begin by setting up some notation. Figure 2 shows how the following objects are situated in the 2-dimensional analogy.

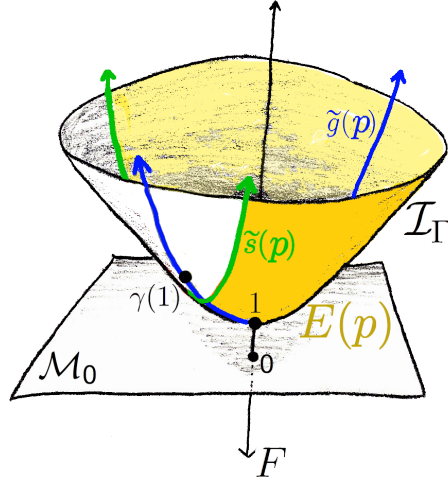
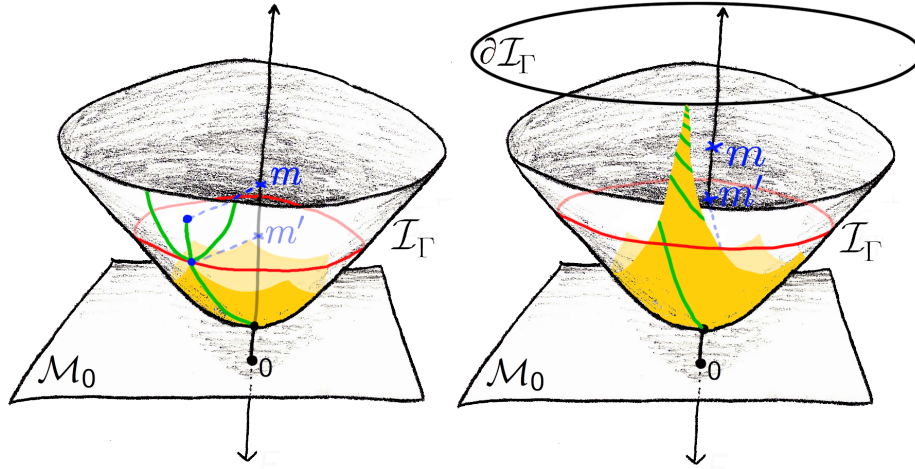
Definition 5.12. Let $\gamma \in \Gamma$ and $p = \gamma(1)$.

- (1) Let $g(p) := g(1, p)$, the geodesic segment from 1 to p , and let $\tilde{g}(p)$ be the complete geodesic containing $g(p)$.
- (2) Let $\tilde{s}(p)$ be the complete geodesic hyperplane perpendicularly bisecting $g(p)$.
- (3) Let $E(p) \subset \mathcal{I}_{\Gamma}$ be the closed half-space satisfying $\partial E(p) = \tilde{s}(p)$ and $1 \in E(p)$.
- (4) We say that γ *contributes a side* to $\mathcal{D}_{\Gamma}(1)$ if the intersection of $\tilde{s}(p)$ with $\partial\mathcal{D}_{\Gamma}(1)$ is codimension-1, and in this case the *side* contributed to $\mathcal{D}_{\Gamma}(1)$ by γ is $s(p) := \tilde{s}(p) \cap \partial\mathcal{D}_{\Gamma}(1)$. (Hence $\tilde{s}(p)$ is the complete geodesic hyperplane containing $s(p)$.)

The strategy now is to search through the non-empty V_t as t increases from 2, and compute the sides contributed along the way. We show that this gives an effective way of finding all of the sides of $\mathcal{D}_{\Gamma}(1)$.

Theorem 5.13. *Let Γ be given by a finite set of generators in $\text{PSL}_2(\mathbb{C})$. By checking the V_t in order of increasing t for points that contribute sides to $\mathcal{D}_{\Gamma}(1)$, the following will occur.*

- (1) *All of the sides will be found in order of their distance from the center.*
- (2) *There is a computable value $m \in F$ such that $\forall t > m$ and $\forall p \in V_t$, p does not contribute a side to $\mathcal{D}_{\Gamma}(1)$.*

FIGURE 2. Geodesics used to find sides of $\mathcal{D}_\Gamma(1)$.FIGURE 3. Finding m in the compact (left) and cusped (right) cases.

Proof. The points in a given V_t can be found using expressions such as Equation (5.5) and traces of words in the generators for Γ . For $p \in V_t$, we have $d_{\mathcal{I}_\Gamma}(1, p) = \text{arcosh}(\text{tr}(p)/2)$ and by Proposition 5.4, the set $\{t \mid V_t \neq \emptyset\}$ is discrete, thus the sides would be found in order of their distance from (1). Since $\mathcal{D}_\Gamma(1)$ has finitely many sides, this proves (1).

To prove (2) define

$$\mathcal{R}_u := \bigcap_{\substack{t \leq u \\ p \in V_t}} E_p.$$

Then \mathcal{R}_u is the region containing 1 delineated by all sides contributed up to trace u . When u is large enough that these sides include all the sides of $\mathcal{D}_\Gamma(1)$, we have $\mathcal{R}_u = \mathcal{D}_\Gamma(1)$, so it suffices to give an algorithm to determine when this occurs.

Suppose Γ is cocompact. Then $\mathcal{D}_\Gamma(1)$ is compact, thus there will be a minimal $m' \in K$ so that $\mathcal{R}_{m'}$ is a compact region. To detect when this occurs, compute the circles that the $\tilde{s}(p)$ approach on $\partial\mathcal{I}_\Gamma$ (this can be aided using the slope of $g(p)$ from Definition 5.7). As soon as every point of $\partial\mathcal{I}_\Gamma$ is contained in the interior of one of these circles, let m' be the most recent trace searched and then $\mathcal{R}_{m'}$ will have enclosed a compact region. Now $\mathcal{R}_{m'}$ is not necessarily $\mathcal{D}_\Gamma(1)$, but from here we can compute a trace m sufficiently high to check up to so that $\mathcal{R}_m = \mathcal{D}_\Gamma(1)$. Let $m = m'^2$, then by Proposition 5.6, $d_{\mathcal{I}_\Gamma}(1, m) = 2d_{\mathcal{I}_\Gamma}(1, m')$, so that for any point p with $\text{tr}(p) > m$, the lowest point on $\tilde{s}(p)$ will be above \mathcal{R}_m , as illustrated on the left side of Figure 3 (in the 2-dimensional analogy). As we check points up to m , it is possible that new sides will be contributed, truncating $\mathcal{R}_{m'}$, but after trace m this becomes impossible.

Suppose Γ is non-cocompact. Then $\mathcal{D}_\Gamma(1)$ will not be a region of bounded trace but instead a non-compact region in \mathcal{I}_Γ together with some finite number of points on $\partial\mathcal{I}_\Gamma$, each having a cuspidal neighborhood in \mathcal{I}_Γ homeomorphic to $T^2 \times [0, \infty)$. There will be a first $m' \in K$ where $\mathcal{R}_{m'}$ takes this same form, computable as follows. Check the boundary circles approached by the $\tilde{s}(p)$, as in the compact case, but now look for when the only points not included in the interiors of the circles are some finitely many points q where four bisectors $\tilde{s}(p_1), \tilde{s}(p_2), \tilde{s}(p_3), \tilde{s}(p_4)$ intersect (labeled by adjacency). When that happens, compute for each q the pair of parabolic isometries pairing opposite sides $s(p_1), s(p_3)$ and $s(p_2), s(p_4)$, and check whether those lie in the group (see Remark 5.14). If so, a neighborhood of q forms a cusp of the manifold, and will not be truncated by any future $\tilde{s}(p)$. If not, continue finding sides as before until it does. There is a computable neighborhood $N(q)$ of q so that any geodesic $\tilde{g}(p)$ entering $N(q)$ would converge to $[q]$ in the quotient $\mathcal{I}_\Gamma/\Gamma$ [2]. Therefore, all future $\tilde{g}(p)$ must avoid $N(q)$. When this happens, let m' be the supremum of traces of points on $\left(\bigcup_{\ell=1}^4 \tilde{s}(p_\ell)\right) \setminus N(p)$. The right side of Figure 3 illustrates this in the case where there is a single cusp (in the 2-dimensional analogy), and in that picture $N(p)$ is the yellow cuspidal region above the red circle. Once every cuspidal neighborhood of each point like q has been found, let m'' be the supremum among all the respective m' , or the maximal height in the compact portion if that is higher. Then let $m = m''^2$ and proceed as in the compact case. \square

We now outline the algorithm in a manner which lends itself to computer implementation.

- (1) Obtain a set of matrix generators for $\Gamma \cong \pi_1(X)$ so that $B\Gamma$ is a normalized Macfarlane quaternion algebra. Let $t_0 = 2$.
- (2) Find $t := \min \{\text{tr}(p) \neq t_0 \mid p \in O_\Gamma\}$, then find the elements of V_t .
- (3) For each $p \in V_t$, compute $\tilde{s}(p)$. Then determine whether \mathcal{R}_t encloses a region as described in the proof (up to some boundary points in the non-compact case). If not, replace t_0 with t and return to step (2). If so, continue to step (4).
- (4) Find the value m as in the proof so that if $\text{tr}(\gamma(1)) > m$, then $\tilde{s}(\gamma)$ does not intersect \mathcal{R}_m . If $t \leq m$, replace t_0 with t and return to step (2). If $t > m$, then continue to step (5).
- (5) For each side $s(\gamma)$, compute $\gamma^{-1}(b(\gamma))$. The intersection of this with ∂R will be some other side s , and gives a side-pairing $s(\gamma) \sim s$.

Remark 5.14. It is always possible to check, given a set of generators, whether some element of $B\Gamma$ is a member of Γ [10]. The methods differ depending on the complexity of Γ and on the number of generators.

There are analogous results to all of the above for hyperbolic surfaces, which we apply below.

5.5. A Hyperbolic Punctured Torus. Non-compact arithmetic examples such as this one make for a straightforward application of Diophantine arithmetic. Let S be the hyperbolic punctured torus. Then $\pi_1(S)$ can be represented by $\Delta < \mathrm{PSL}_2(\mathbb{Z})$ where Δ is the torsion-free subgroup of the modular group. Then $\Delta = \langle \gamma, \delta \rangle$ where

$$\gamma = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \delta = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} [4].$$

The quaternion algebra of Δ is $B\Delta = \left(\frac{1,1}{\mathbb{Q}}\right) = \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j \oplus \mathbb{Q}ij$ where $i^2 = j^2 = 1$ and $ij = -ji$, and let $\mathcal{A} = B\Delta$. Then \mathcal{A} contains the restricted Macfarlane space $\mathcal{L} = \mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j$, and $\mathcal{I}_\Delta := \mathcal{L}_+^1$ is a quaternion hyperboloid model for S .

Recall that $\mathcal{A} \cong \mathrm{M}_2(\mathbb{Q})$. So the map (5.8) gives the \mathbb{Q} -algebra isomorphisms

$$\begin{aligned} \rho_{\mathcal{A}} : \mathcal{A} &\rightarrow \mathrm{M}_2(\mathbb{Q}), \quad w + xi + yj + zij \mapsto \begin{pmatrix} w-x & y-z \\ w+z & y+z \end{pmatrix}, \\ \rho_{\mathcal{A}}^{-1} : \mathrm{M}_2(\mathbb{Q}) &\rightarrow \mathcal{A}, \quad \begin{pmatrix} s & t \\ u & v \end{pmatrix} \mapsto \frac{v+s}{2} + \frac{v-s}{2}i + \frac{u+t}{2}j + \frac{u-t}{2}ij, \end{aligned}$$

the map (5.9) gives the isometry

$$\iota_\Delta : \mathcal{I}_\Delta \rightarrow \mathcal{H}^2, \quad w + xi + yj \mapsto \frac{y + J}{w + x},$$

and these transfer the Macfarlane model to the Möbius action of Δ on \mathcal{H}^2 . When the context is clear, we use $\Delta = \langle \gamma, \delta \rangle$ to mean both the matrix group and the corresponding quaternion group under $\rho_{\mathcal{A}}^{-1}$, where

$$\gamma = \frac{3}{2} + \frac{1}{2}i + j \quad \text{and} \quad \delta = \frac{3}{2} + \frac{1}{2}i - j.$$

To implement the algorithm, we want to find the points in $O_\Delta = \mathrm{Orb}_\Delta(1)$ in order of increasing trace. Since Δ consists of all the non-elliptic elements of $\mathrm{PSL}_2(\mathbb{Z})$, Δ is closed under transposition, therefore $\forall q \in \Delta, qq^\dagger \in \Delta$, which implies $\mathrm{Orb}_\Delta(1) \subset \Delta$. Then $\forall t \in \mathbb{N}$

$$V_t \subset \{q \in \Delta \cap \mathcal{I}_\Delta \mid \mathrm{tr}(q) = t\}.$$

But using $n(\Delta \cap \mathcal{I}_\Delta) = 1$, and the fact that hyperbolic elements have traces in $\mathbb{R} \setminus [-2, 2]$, we can characterize these elements using a Diophantine equation where the only solutions for t lie in $\mathbb{Z} \setminus \{0, \pm 1\}$. In particular,

$$\{q \in \Delta \cap \mathcal{I}_\Delta \mid \mathrm{tr}(q) = t\} = \left\{ \frac{t}{2} + \frac{t-2x}{2}i + y \mid x^2 + y^2 = tx - 1, x, y \in \mathbb{Z} \right\}.$$

Once we know what this (finite) set is, we can find V_t by checking whether each element can be written in the form $q = ww^\dagger$, for some word w in the generators of Δ . If it can, we get that $\tilde{s}(q)$ (in the notation of Definition 5.12) passes halfway between 1 and q . If it cannot, we get that $q^2 \in \mathrm{Orb}_\Delta(1)$ and $\tilde{s}(q^2)$ passes through q , by Proposition 5.6.

Table 1 lists some data from implementing this process, giving the points in $\mathcal{I}_\Delta \cap \Delta$ up to trace 18. Points in $\mathrm{Orb}_\Delta(1)$ that contribute sides to $\mathcal{D}_\Delta(1)$ are in bold. For each $p \in \mathcal{I}_\Delta \cap \Delta$, the direction of the corresponding geodesic ray is given by a normalized representative of the slope of p (in the sense of Definition 5.7). In the rightmost column, the corresponding points in \mathcal{H}^2 under ι_Δ are given. Notice that (as predicted by Proposition 5.6), if a point p in the chart does not lie in $\mathrm{Orb}_\Delta(1)$, then later the point p^2 does, and has the same slope. For example, the elements of $\mathcal{I}_\Delta \cap \Delta$ at trace 3 lead to sides contributed at trace 6.

Figure 4 shows in \mathcal{H}^2 which sides are contributed at each trace until $\mathcal{D}_\Delta(1)$ is complete, and illustrates how the induced side-pairings create a punctured torus.

5.6. Non-compact Arithmetic Hyperbolic 3-Manifolds. As mentioned in §4.2.1, a manifold in this class has a fundamental group which can be represented by a torsion-free finite index subgroup of a Bianchi group $\mathrm{PSL}_2(\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})})$, $d \in \mathbb{N}$ (square-free). Let Γ be such a group where $B\Gamma = A\Gamma$. Then $B\Gamma$ is Macfarlane and we can find a Dirichlet domain for Γ by a similar method to that used in the previous subsection.

Since the only real traces occurring in $\mathrm{PSL}_2(\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})})$ lie in \mathbb{Z} , the V_t can only be non-empty when $t \in \{2, 3, 4, \dots\}$. Since $\mathrm{PSL}_2(\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})})$ is closed under complex conjugation, $O_\Gamma \subset \mathrm{PSL}_2(\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})})$, so like in the previous example $V_t \subset \{\gamma \in \mathrm{PSL}_2(\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})}) \mid \mathrm{tr}(\gamma) = t\}$. The points in $\mathrm{PSL}_2(\mathbb{Z}_{\mathbb{Q}(\sqrt{-d})})$ of trace t correspond to solutions to the Diophantine equation arising from $\det \begin{pmatrix} r & s \\ \bar{s} & t - r \end{pmatrix} = 1$ where $r \in \mathbb{Z}$ and $s \in \mathcal{O}_d$. Once we find those, we can use the generators for Γ to determine which of them make up V_t .

Example 5.15. The fundamental group of the Whitehead Link Complement can be represented by the finite-index subgroup of $\mathrm{PSL}_2(\mathbb{Z}[\sqrt{-1}])$ generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & \sqrt{-1} \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ -1 - \sqrt{-1} & 1 \end{pmatrix}$ [16]. Suppressing some details, a computer implementation of the process described above yields the Dirichlet domain for this group illustrated in Figure 5, where we view the faces from above in \mathcal{H}^3 .

6. ACKNOWLEDGEMENTS

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TABLE 1. A punctured torus group intersected with its own quaternion hyperboloid model.

trace	$q \in \mathcal{I}_\Delta \cap \Delta$	slope of q	$\rho_\Delta(q) \in \mathrm{PSL}_2(\mathbb{Z})$	$\iota_\Delta(q) \in \mathcal{H}^2$
2	1	–	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	J
3	$\frac{3}{2} + \frac{1}{2}i \pm j$	$(1, \pm 2)$	$\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 2 \end{pmatrix}$	$\pm \frac{1}{2} + \frac{1}{2}J$
	$\frac{3}{2} - \frac{1}{2}i \pm j$	$(-1, \pm 2)$	$\begin{pmatrix} 2 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}$	$\pm 1 + J$
6	$3 + 2i \pm 2j$	$(1, \pm 1)$	$\begin{pmatrix} 1 & \pm 2 \\ \pm 2 & 5 \end{pmatrix}$	$\pm \frac{2}{5} + \frac{1}{5}J$
	$\mathbf{3} - \mathbf{2}i \pm \mathbf{2}j$	$(-1, \pm 1)$	$\begin{pmatrix} 5 & \pm 2 \\ \pm 2 & 1 \end{pmatrix}$	$\pm \mathbf{2} + J$
7	$\frac{7}{2} + \frac{3}{2}i \pm \mathbf{3}j$	$(1, \pm 2)$	$\begin{pmatrix} 2 & \pm 3 \\ \pm 3 & 5 \end{pmatrix}$	$\pm \frac{3}{5} + \frac{1}{5}J$
	$\frac{7}{2} - \frac{3}{2}i \pm 3j$	$(-1, \pm 2)$	$\begin{pmatrix} 5 & \pm 3 \\ \pm 3 & 2 \end{pmatrix}$	$\pm \frac{3}{2} + \frac{1}{2}J$
11	$\frac{11}{2} + \frac{9}{2}i \pm \mathbf{3}j$	$(3, \pm 2)$	$\begin{pmatrix} 1 & \pm 3 \\ \pm 3 & 10 \end{pmatrix}$	$\pm \frac{3}{10} + \frac{1}{10}J$
	$\frac{11}{2} - \frac{9}{2}i \pm 3j$	$(-3, \pm 2)$	$\begin{pmatrix} 10 & \pm 3 \\ \pm 3 & 1 \end{pmatrix}$	$\pm 3 + J$
15	$\frac{15}{2} + \frac{11}{2}i \pm 5j$	$(11, \pm 10)$	$\begin{pmatrix} 2 & \pm 5 \\ \pm 5 & 13 \end{pmatrix}$	$\pm \frac{5}{13} + \frac{1}{13}J$
	$\frac{15}{2} - \frac{11}{2}i \pm \mathbf{5}j$	$(-11, \pm 10)$	$\begin{pmatrix} 13 & \pm 5 \\ \pm 5 & 2 \end{pmatrix}$	$\pm \frac{5}{2} + \frac{1}{2}J$
	$\frac{15}{2} + \frac{5}{2}i \pm 7j$	$(5, \pm 14)$	$\begin{pmatrix} 5 & \pm 7 \\ \pm 7 & 10 \end{pmatrix}$	$\pm \frac{7}{10} + \frac{1}{10}J$
	$\frac{15}{2} - \frac{5}{2}i \pm 7j$	$(-5, \pm 14)$	$\begin{pmatrix} 10 & \pm 7 \\ \pm 7 & 5 \end{pmatrix}$	$\pm \frac{7}{5} + \frac{1}{5}J$
18	$9 + 8i \pm 4j$	$(2, \pm 1)$	$\begin{pmatrix} 1 & \pm 4 \\ \pm 4 & 17 \end{pmatrix}$	$\pm \frac{4}{17} + \frac{1}{17}J$
	$9 - 8i \pm 4j$	$(-2, \pm 1)$	$\begin{pmatrix} 17 & \pm 4 \\ \pm 4 & 1 \end{pmatrix}$	$\pm 4 + J$
	$9 + 4i \pm 8j$	$(1, \pm 2)$	$\begin{pmatrix} 5 & \pm 8 \\ \pm 8 & 13 \end{pmatrix}$	$\pm \frac{8}{13} + \frac{1}{13}J$
	$9 - 4i \pm 8j$	$(-1, \pm 2)$	$\begin{pmatrix} 13 & \pm 8 \\ \pm 8 & 5 \end{pmatrix}$	$\pm \frac{8}{5} + \frac{1}{5}J$

FIGURE 4. Dirichlet domain for a hyperbolic punctured torus.

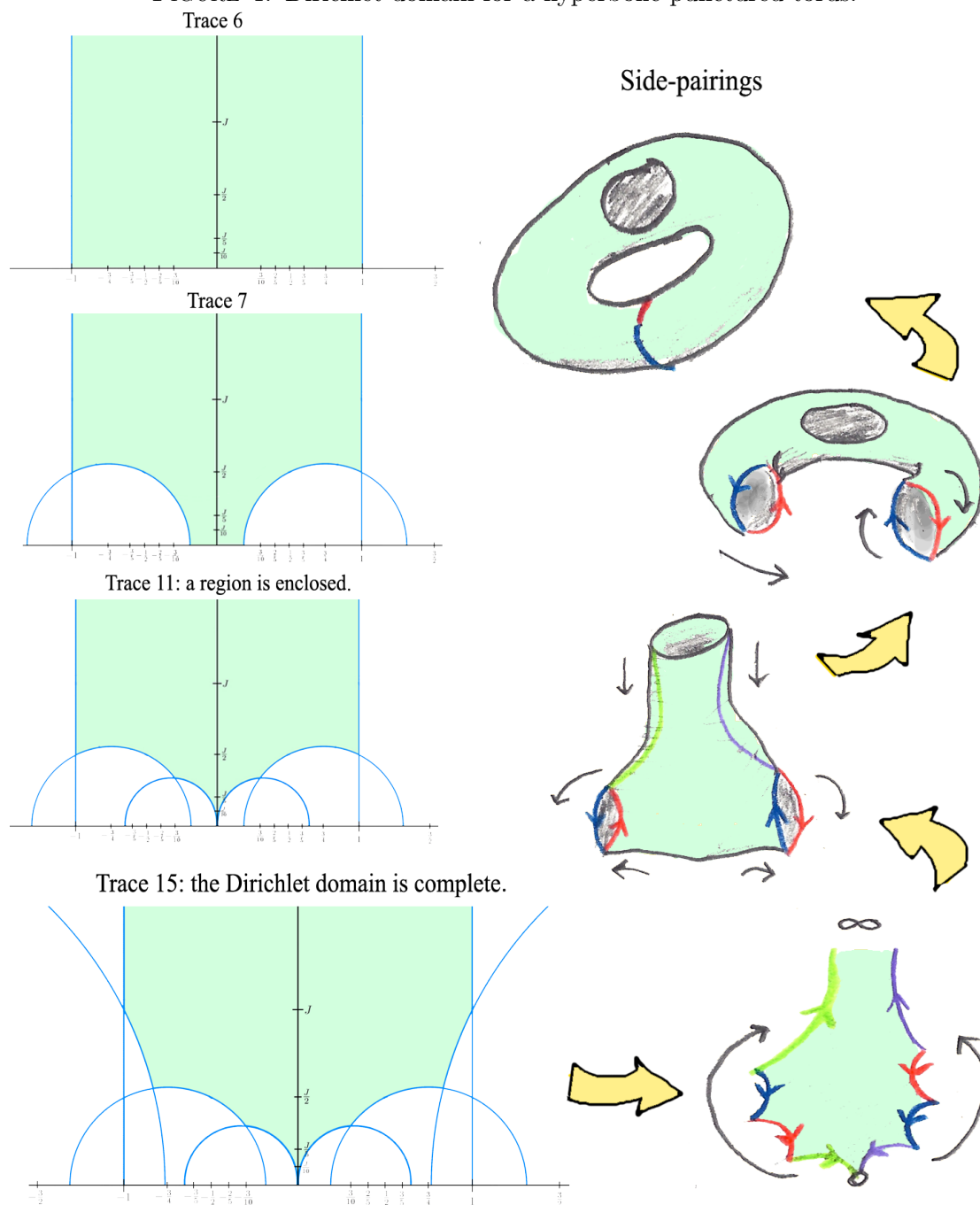
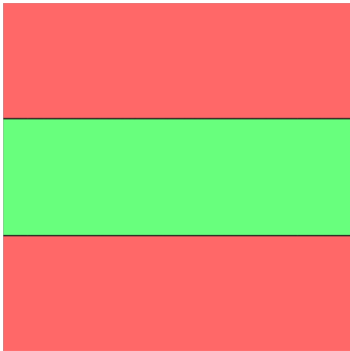
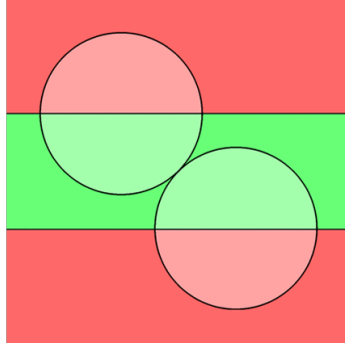
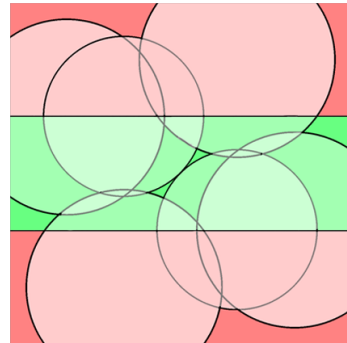
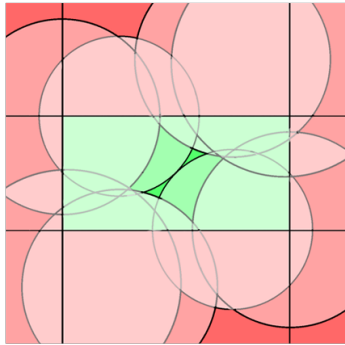
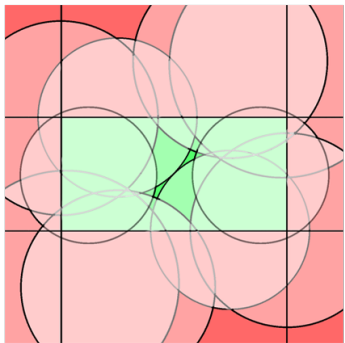
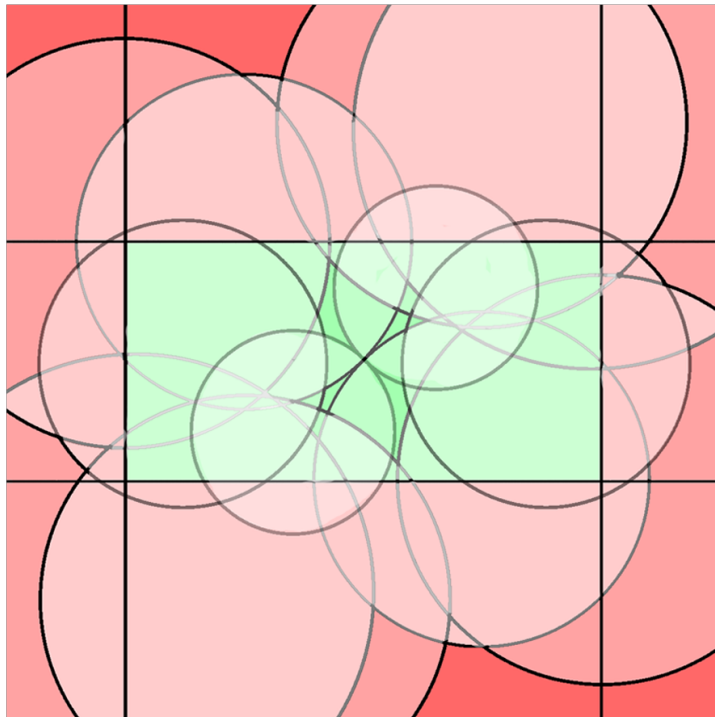


FIGURE 5. Dirichlet domain for the Whitehead link complement.

**Trace 3:** a pair of parallel half-planes.**Trace 4:** a pair of hemispheres tangent at zero.**Trace 5:** four overlapping hemispheres symmetric around the vertical axis.**Trace 6:** two parallel half-planes completing a cusp at infinity, & and two larger hemispheres..**Trace 7:** two smaller hemispheres centered on the real axis further truncate the region.**The completed Dirichlet domain.** At trace 10 a pair of hemispheres tangent at zero completes the cusp suggested at trace 4. Several other orbit points, starting at trace 6, are not included because they are too far away to contribute sides. After trace 10, this is true for all further orbit points.